Remark 2. Condition (13) in Theorem II can also be independently obtained using the results of Grujic and Siljak[6] for interconnected systems.

Let $V_l(x_l) = \sqrt{x_l^T P_l x_l}$ be a Lyapunov-type function for the uncoupled subsystem 1. It can be shown (in a way similar to the proof of the lemma) that

$$\sqrt{\lambda_m(P_l)}\,\|x_l\| \le V_l(x_l) \le \sqrt{\lambda_M(P_l)}\,\|x_l\|$$

$$\dot{V}_l(x_l) \le -\tfrac{1}{2}\,\frac{\lambda_m(Q_l)}{\sqrt{\lambda_M(P_l)}}\,\|x_l\|$$

$$\left(\frac{\partial V_l}{\partial x_l}\right)^T \alpha x_r \le \sqrt{\lambda_M(P_l)}\,\|\alpha\|_s\,\|x_r\|$$

Similar inequalities can be obtained for the residual subsystem assuming

$$V_2(x_r) = \sqrt{x_r^T P_r x_r}$$

Theorem 1 of Ref. 6 can now be applied to obtain Eq. (13).

Discussion and Numerical Results

Theorems I and II, proved above, give two different sufficient conditions for stability. However, the following drawbacks are normally associated with Lyapunov mehods: 1) the conditions obtained are conservative, and 2) considerable arbitrariness is involved in the selection of Lyapunov functions. The choice of matrices Q, Q_l and Q_r in Eqs. (7), (8), and (9) is arbitrary, the only constraint being that they are positive definite. More investigation is needed in the selection of these matrices in order to obtain the least conservative bounds.

A rigid-body-plus-seven-mode, normal coordinate, planar model of a uniform free-free beam was chosen for evaluating the stability bounds (for a development of the equations for a uniform free-free beam, see Ref. 7). Bending mode frequencies and assumed damping ratios of the beam are given in Table 1. One torque actuator, one attitude sensor and one rate sensor were used. Bounds β_l and β_2 were computed with matrices Q, Q_l and Q_r equal to identity matrices of appropriate dimensions, for a nominal set of LQG regulator and estimator gains (G and K). The regulator was designed to control rigid-body-plus-first two modes, and the estimator was designed to estimate rigid-body-plus-first five modes. For this case, β_l was 0.184×10^{-7}, and β_2 was 0.553×10^{-5}. Thus, bound β_2 is far less conservative than bound β_l (by a factor of about 300). In fact, if Q is block-diagonal, the 1-system and r-system give uncoupled Lyapunov matrix equations. In this case, it can be seen from Eqs. (10) and (13), that $\beta_l \le \beta_2$. In addition, Eq. (13) is consistent with the fact that the absence of either spillover term assures stability. Thus, Eq. (13) appears to be a better sufficient condition.

Conclusions

Two sufficient conditions were derived via Lyapunov methods for asymptotic stability of large space structures using a class of reduced-order controllers. These conditions give allowable bounds on the spectral norms of control and observation "spillover" terms. The sufficient condition given in Eqs. (13) appears to be less conservative, and should be useful as a design tool for the control of large space structures.

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The Orientation Vector Differential Equation

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Introduction

THE orientation or Euler vector $\psi(t)$ relates a rotating body frame to a reference frame at any time t, in that rotating a reference frame about an axis coincident with $\psi(t)$ through an angle which is equal to the magnitude of $\psi(t)$ will bring the reference frame into coincidence with the body frame. Thus, if we define θ as the magnitude of the rotation and u as the unit vector about which the rotation has occurred, then $\psi(t) = \theta(t)u(t)$.

In Ref. 1 Bortz derived an expression for the time rate of change of the orientation vector and discussed some of its applications. The vector $\dot{\psi}(t)$ was shown to have two components: $\omega(t)$, the component due to inertially measurable angular motion (angular velocity vector), and $\dot{\sigma}(t)$, the component due to noninertially measurable angular motion (noncommutativity rate vector), where $\psi(t)$ and $\dot{\sigma}(t)$ are orthogonal. The latter component reflects the fact that the orientation of a rigid body after a sequence of rotations depends on the order of the rotations as well as the magnitude, and axis orientation of the individual rotations.

While Bortz's derivation is on the whole straightforward, the derivation is also rather lengthy and involves a number of algebraic manipulations. In this paper we give a derivation of $\dot{\psi}(t)$ which we believe to be much simpler. Most of the simplification comes from exploiting a kinematical property of the Euler-Rodrigues rotation parameters. This approach was not used in Ref. 1. The three-parameter Euler-Rodrigues method is a special case of the more widely known four-parameter Euler method.[2]

Derivation

Before proving the main result, we first define some terms and then state two lemmas which will help expedite the derivation of $\dot{\psi}(t)$.

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With regard to notation, let $\omega(t)$ and the skewsymmetric matrix $\boldsymbol{\Omega}(t)$ be related in the usual manner; i.e.

$$[\omega_1\omega_2\omega_3]' = \boldsymbol{\omega} \longleftrightarrow \boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Also, the vector ψ and the skewsymmetric matrix $\boldsymbol{\Psi}$ are similarly related.

Let $\boldsymbol{C}(t)$ denote the direction cosine matrix which transforms body frame vectors to the reference frame and let $\boldsymbol{v}(t)$ be defined as follows:

$$\boldsymbol{v} = \begin{bmatrix} C_{32} - C_{23} \\ C_{13} - C_{31} \\ C_{21} - C_{12} \end{bmatrix}$$

The matrix $\boldsymbol{C}(t)$ is related, of course, to $\boldsymbol{\Omega}(t)$ via the equation $\dot{\boldsymbol{C}}(t) = \boldsymbol{C}(t)\boldsymbol{\Omega}(t)$.

Finally, $|\psi| = \sqrt{\psi'\psi} = \theta$ since $\boldsymbol{u}(t)$ is a unit vector.

We are now in a position to state two well-known results; viz.

Lemma 1

$$2\cos(\theta) = \mathrm{tr}(C) - 1 \tag{1}$$

$$2\,u\sin(\theta) = v \tag{2}$$

Lemma 2

$$\mathrm{tr}(\boldsymbol{C}\boldsymbol{\Omega}) = -\boldsymbol{\omega}'\boldsymbol{v}$$

A proof of Lemma 1 can be found in Ref. 3 while Lemma 2 is obvious.

We now prove the main result.

Theorem

$$\dot{\psi}(t) = \omega(t) + \dot{\sigma}(t)$$

where

$$\dot{\sigma}(t) = \boldsymbol{\Psi}(t)\omega(t)/2 + [2 - |\psi(t)|\cot(|\psi(t)|/2)] \times \boldsymbol{\Psi}^2(t)\omega(t)/2|\psi(t)|^2$$

and $\psi'(t)\dot{\sigma}(t) = 0$ for each t.

Proof. Since $\dot{\psi} = \dot{u}\theta + u\dot{\theta}$, we need only to obtain expressions for $\dot{u}$ and $\dot{\theta}$ to complete the derivation. To obtain the latter variable note that differentiating Eq. (1) yields

$$\dot{\theta} = \frac{-\mathrm{tr}(\dot{\boldsymbol{C}})}{2\sin(\theta)} = \frac{-\mathrm{tr}(\boldsymbol{C}\boldsymbol{\Omega})}{2\sin(\theta)} = \frac{\boldsymbol{\omega}'\boldsymbol{v}}{2\sin(\theta)} = \boldsymbol{\omega}'\boldsymbol{u} \tag{3}$$

where the last two equalities follow from Lemma 2, and Eq. (2) of Lemma 1, respectively. The most direct approach for obtaining an expression for $\dot{u}$ is via the kinematical differential equations associated with the Euler-Rodrigues parameters. Namely,

$$\dot{q} = (I + qq' + Q)\omega/2 \tag{4}$$

where $q = u\tan(\theta/2)$ and Q is the 3×3 skew symmetric matrix whose elements are the components of q. Equation (4) was first derived by Cayley (circa 1843), though Roberson[2] has given a modern derivation. Carrying out the indicated differentiation on the left side of Eq. (4), we have

$$\dot{u}\tan(\theta/2) + u[1 + \tan^2(\theta/2)]\dot{\theta}/2 = [I + uu'\tan^2(\theta/2) + U\tan(\theta/2)]\omega/2$$

Transposing terms, substituting for $\dot{\theta}$, and making use of the identity $U^2 = uu' - I$ immediately yields

$$\dot{u} = U\omega/2 - \cot(\theta/2)\,U^2\omega/2 \tag{5}$$

By using Eqs. (3) and (5) we can now write $\dot{\psi}$ as

$$\dot{\psi} = U\omega\theta/2 - \theta\cot(\theta/2)\,U^2\omega/2 + uu'\omega$$

Adding and subtracting ω, noting that $\boldsymbol{\Psi} = U\theta$, and recalling that $|\psi| = \theta$ yields the desired result:

$$\dot{\psi} = \omega + \boldsymbol{\Psi}\omega/2 + [2 - |\psi|\cot(|\psi|/2)]\boldsymbol{\Psi}^2\omega/2|\psi|^2$$

That $\psi'\dot{\sigma} = 0$ (i.e., ψ and $\dot{\sigma}$ are orthogonal) follows from the fact the $\boldsymbol{\Psi}\psi = \boldsymbol{0}$, which is true for any three-dimensional vector ψ. Q.E.D.

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